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SCATTERING FOR SMALL ENERGY SOLUTIONS OF NLS WITH PERIODIC POTENTIAL IN 1D

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Abstract

Given $H \equiv -\partial_x^2 + V(x)$ with $V : \mathbb{R} \to \mathbb{R}$ a smooth periodic potential, for $\mu \in \mathbb{R}\backslash\{0\}$ and $p \geq 7$, we prove scattering for small solutions to

$$i\partial_t u + Hu = \mu |u|^{p-1}u,\ (t,x) \in \mathbb{R}\times\mathbb{R}, \quad u(0) = u_0 \in H^1(\mathbb{R}).$$

1. Introduction

In this paper, for $\beta : \mathbb{R}^+ \to \mathbb{R}$ a suitable nonlinearity, we prove scattering of small solutions of

$$i\partial_t u + Hu = \beta(|u|^2)u,\ (t,x) \in \mathbb{R}\times\mathbb{R}, \quad u(0) = u_0 \in H^1(\mathbb{R}) \tag{1.1}$$

where $H \equiv -\partial_x^2 + V(x)$ with $V(x)$ a smooth real valued periodic potential. To do this we need to write appropriate Strichartz estimates for H. For every $1 \leq p, q \leq \infty$ we consider the Birman-Solomjak spaces

$$l^p(\mathbb{Z}, L_t^q[n,n+1]) \equiv \left\{ f \in L^q_{loc}(\mathbb{R}) \text{ s.t. } \{\|f\|_{L^q[n,n+1]}\}_{n\in\mathbb{Z}} \in l^p(\mathbb{Z}) \right\}, \tag{1.2}$$

endowed with the natural norms

$$\|f\|^p_{l^p(\mathbb{Z},L_t^q[n,n+1])} \equiv \sum_{n\in\mathbb{Z}} \|f\|^p_{L_t^q[n,n+1]} \ \forall\ 1 \leq p < \infty \text{ and } 1 \leq q \leq \infty$$

$$\|f\|_{l^\infty(\mathbb{Z},L_t^q[n,n+1])} \equiv \sup_{n\in\mathbb{Z}} \|f\|_{L^q[n,n+1]}.$$

We consider the Sobolev spaces

$$W^{k,p}(\mathbb{R}) \equiv \{f \in \mathcal{S}'(\mathbb{R}) | (1-\partial_x^2)^{k/2} f \in L^p(\mathbb{R})\}. \tag{1.3}$$

For $p = 2$ we set $H^k(\mathbb{R}) \equiv W^{k,2}(\mathbb{R})$. Then we prove:

Theorem 1.1. *Assume $\beta(t) \in C^3(\mathbb{R},\mathbb{R}^3)$ with $\beta(0) = \beta'(0) = \beta''(0) = 0$ and that $V(x)$ is a smooth periodic and nonconstant real valued potential. Then there exists $\epsilon_0 > 0$ such that for any initial data $u_0 \in H^1(\mathbb{R})$ with $\|u_0\|_{H^1(\mathbb{R})} < \epsilon_0$ problem (1.1) is globally well–posed. Moreover there exists $C = C(\epsilon_0) > 0$ such that it is possible to split $u(t,x) = u_1(t,x) + u_2(t,x)$ so that for any couple (r,p) that satisfies*

$$2/r + 1/p = 1/2 \text{ and } (r,p) \in [4,\infty]\times[2,\infty], \tag{1.4}$$

we have

$$\|u_1(t,x)\|_{\ell^{\frac{3}{2}r}(\mathbb{Z},L_t^\infty([n,n+1],W^{1,p}(\mathbb{R})))} + \|u_2(t,x)\|_{L_t^r(\mathbb{R},W^{1,p}(\mathbb{R}))} \le C\|u_0\|_{H^1(\mathbb{R})}. \tag{1.5}$$

Furthermore, there exist $u_\pm \in H^1(\mathbb{R})$ *with* $\|u_\pm\|_{H^1(\mathbb{R})} < C\|u_0\|_{H^1(\mathbb{R})}$ *such that*

$$\lim_{t\to\pm\infty} \|u(t,x) - e^{-itH}u_\pm\|_{H^1(\mathbb{R})} = 0. \tag{1.6}$$

If $V(x)$ is constant there is a considerable literature on (1.1). A basic tool are the Strichartz estimates, see [1, 3], which follow, for $\mathcal{V}(t) \equiv e^{it\partial_x^2}$, from

$$\|\mathcal{V}(t)f\|_{L^\infty(\mathbb{R})} \le C|t|^{-\frac{1}{2}}\|f\|_{L^1(\mathbb{R})}. \tag{1.7}$$

For any $V(x)$ not constant (1.7) is not true and by [2] we have instead

$$\|e^{itH}f\|_{L^\infty(\mathbb{R})} \le CMax\{|t|^{-\frac{1}{2}}, \langle t\rangle^{-\frac{1}{3}}\}\|f\|_{L^1(\mathbb{R})}. \tag{1.8}$$

(1.8) requires a new set of Stricharz estimates for e^{itH}. This is done in the next section. In the subsequent section we apply the Stricharz estimates to the nonlinear problem.

In the sequel we shall use the following notations:

$$L_x^p = L^p(\mathbb{R}_x), W_x^{k,p} = W^{k,p}(\mathbb{R}_x), H_x^s = H^s(\mathbb{R}_x).$$

2. Stricharz estimates

For any $r \in [1,\infty]$ we set $r' = \frac{r}{r-1}$. By standard arguments it is possible to prove:

Lemma 2.1. *Let* $\mathcal{U}(t) : L_x^2 \to L_x^2$ *be a uniformly bounded group in* L_x^2 *such that* $\|\mathcal{U}(t)f\|_{L_x^\infty} \le C_1\langle t\rangle^{-\frac{1}{3}}\|f\|_{L_x^1}$. *Then there exists* $C > 0$ *such that for every pair which satisfies* (1.4) *we have*

$$\|\mathcal{U}(t)f\|_{\ell^{\frac{3}{2}r}(\mathbb{Z},L_t^\infty([n,n+1],L_x^p))} \le C\|f\|_{L_x^2}. \tag{2.1}$$

Moreover there is $C > 0$ *such that for any two pairs* (r_1,p_1) *and* (r_2,p_2) *that satisfy* (1.4) *we have*

$$\begin{aligned}\left\|\int_0^t \mathcal{U}(t-s)F(s)ds\right\|_{\ell^{\frac{3}{2}r_1}(\mathbb{Z},L_t^\infty([n,n+1],L_x^{p_1}))} \\ \le C\|F\|_{\ell^{(\frac{3}{2}r_2)'}(\mathbb{Z},L_t^1([n,n+1],L_x^{p_2'}))}.\end{aligned} \tag{2.2}$$

Our next step is:

Lemma 2.2. *There exists a projection* $\pi : L_x^2 \to L_x^2$ *which commutes with* e^{itH} *such that the group* $\mathcal{U}(t) \equiv \pi e^{itH}$ *satisfies the hypotheses of Lemma 2.1 and the group* $\mathcal{V}(t) \equiv (1-\pi)e^{itH}$ *satisfies the estimate* (1.7).

Proof. We have $e^{itH}(x,y) = K(t,x,y)$

$$K(t,x,y) = \int_{\mathbb{B}} e^{i(tE(k)-(x-y)k)}m_-^0(x,k)m_+^0(y,k)dk$$

with $e^{\mp ixk}m_\mp^0(x,k)$ the Bloch functions and $E(k)$ the band function, see [2]. By §4 [2] there are two characteristic functions $\chi_j(k)$, $j = 1,2$ such that $1 = \chi_1(k)+\chi_2(k)$ in $\mathbb{R}$ and such that, if we set

$$K_j(t,x,y) = \int_{\mathbb{R}} e^{i(tE(k)-(x-y)k)} m_-^0(x,k) m_+^0(y,k) \chi_j(k) dk,$$

then there is a fixed $C > 0$ such that $|K_1(t,x,y)| \le C\langle t\rangle^{-\frac{1}{3}}$ and $|K_2(t,x,y)| \le C|t|^{-\frac{1}{2}}$ for all $(t,x,y) \in \mathbb{R}^3$. Notice that [2] treats the generic case when all the spectral gaps of the spectrum $\sigma(H)$ of H are nonempty, but the arguments are the same in the case $\sigma(H)$ has infinitely many bands with some empty gaps, and much easier if $\sigma(H)$ has finitely many bands.

3. Proof of theorem 1.1

The global well posedness in H_x^1 is well know since it follows from standard theory. Specifically, following a sequence of arguments in [1] one has:

(1) if $\|u_0\|_{H_x^1} < \epsilon \le \epsilon_0$ with ϵ_0 sufficiently small, (1.1) admits a solution $u(t) \in L_t^\infty(\mathbb{R}, H_x^1) \cap W_t^{1,\infty}(\mathbb{R}, H_x^{-1})$;
(2) the above solution is unique;
(3) the solution $u(t)$ can be written in the form

$$u(t) = e^{-itH}u_0 + v(t) \text{ with } v(t) = -i\int_0^t e^{-i(t-s)H}\beta(|u(s)|^2)u(s)ds.$$

(4) the above solution is $u(t) \in C^0(\mathbb{R}, H_x^1) \cap C^1(\mathbb{R}, H_x^{-1})$ and the following quantities are conserved:

$$\|u(t)\|_{L_x^2} = \|u_0\|_{L_x^2},$$
$$E(t) = \int_{\mathbb{R}} \left(|\partial_x u(t,x)|^2 - V(x)|u(t,x)|^2 + 2F(|u(t,x)|^2)\right) dx = E(0)$$

where $F(0) = 0$ and $\partial_{\overline{u}} F(|u|^2) = \beta(|u|^2)u$;
(5) there exists a fixed $C > 0$ such that $\|u(t)\|_{H_x^1} < C\epsilon$ for all $t \in \mathbb{R}$.

Hence we need only to prove the scattering part. By Lemma 2.2 inequality (1.5) is true for some $C = C_0$ for u replaced by $e^{-itH}u_0$. It remains to show that (1.5) is true with u replaced in the left hand side (1.5) by the v in (3). We will show:

Lemma 3.1. *For π the projection in Lemma 2.2, let $v_1(t) = \pi v(t)$ and $v_1(t) = (1-\pi)v(t)$. Then, for any $D > 0$ there are constants $\epsilon_0 > 0$ and $C(D)$ such that if*

$$\|v_1(t,x)\|_{\ell^{\frac{3}{2}r}(\mathbb{Z}, L_t^\infty([n,n+1], W_x^{1,p}))} + \|v_2(t,x)\|_{L_t^r(\mathbb{R}, W_x^{1,p})} \le D\|u_0\|_{H_x^1}$$

for all pairs satisfying (1.4), *and if* $\|u_0\|_{H_x^1} < \epsilon < \epsilon_0$, *then*

$$\|v_1(t,x)\|_{\ell^{\frac{3}{2}r}(\mathbb{Z}, L_t^\infty([n,n+1], W_x^{1,p}))} + \|v_2(t,x)\|_{L_t^r(\mathbb{R}, W_x^{1,p})} \le C(D)\epsilon^6\|u_0\|_{H_x^1}.$$

Proof. We have

$$\begin{aligned} \|v_1\|_{\ell^{\frac{3}{2}r}(\mathbb{Z}, L_t^\infty([n,n+1], W_x^{1,p}))} &\lesssim \|\beta(|u|^2)u\|_{L_t^1(\mathbb{R}, H_x^1)} \lesssim \||u|^6 u\|_{L_t^1(\mathbb{R}, H_x^1)} \\ &\lesssim \|u\|_{L_t^\infty(\mathbb{R}, H_x^1)} \|u\|^6_{L_t^6(\mathbb{R}, L_x^\infty)} \le C\|u_0\|_{H_x^1}\|u\|^6_{L_t^6(\mathbb{R}, L_x^\infty)}. \end{aligned} \tag{3.1}$$

Now we split $u = u_1 + u_2$ setting $u_1(t) = \pi e^{-itH}u_0 + v_1(t)$ and $u_2(t) = (1-\pi)e^{-itH}u_0 + v_2(t)$. Correspondingly we get by hypothesis

$$\|u\|^6_{L_t^6(\mathbb{R}, L_x^\infty)} \lesssim \|u_1\|^6_{\ell^6(\mathbb{Z}, L_t^\infty[n,n+1]), L_x^\infty))} + \|u_2\|^6_{L_t^6 W_x^{1,6}} \le CD^6\|u_0\|^6_{H_x^1}. \tag{3.2}$$

By a similar argument

$$\|v_2\|_{L^r_t W^{1,p}_x} \lesssim \|\beta(|u|^2)u\|_{L^1_t(\mathbb{R},H^1_x)} \lesssim \||u|^6 u\|_{L^1_t(\mathbb{R},H^1_x)} \leq CD^6 \|u_0\|^7_{H^1_x}. \tag{3.3}$$

This yields Lemma 3.1.

The proof of (1.6) is standard and goes as follows.

$$e^{itH}u(t) = u_0 - i\int_0^t e^{isH}\beta(|u(s)|^2)u(s)ds$$

and so for $t_1 < t_2$

$$e^{it_2H}u(t_2) - e^{it_1H}u(t_1) = -i\int_{t_1}^{t_2} e^{isH}\beta(|u(s)|^2)u(s)ds.$$

Then by the proof of Lemma 3.1

$$\begin{aligned} \|e^{it_2H}u(t_2) - e^{it_1H}u(t_1)\|_{H^1_x} &\leq \|\int_{t_1}^{t_2} e^{isH}\beta(|u(s)|^2)u(s)ds\|_{H^1_x} \\ &\leq \|\beta(|u|^2)u\|_{L^1([t_1,t_2],H^1_x)} \to 0 \text{ for } t_1 \to \infty \text{ and } t_1 < t_2. \end{aligned} \tag{3.4}$$

Then $u_+ = \lim_{t\to\infty} e^{itH}u(t)$ satisfies the desired properties. One proves the existence of $u_- = \lim_{t\to-\infty} e^{itH}u(t)$ similarly.

References

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